Journal of Engineering Physics and Thermophysics, Vol. 78, No. 3, 2005

PROCESSES UNDER THE STRESSED-STRAINED STATE OF MEDIA

FLAT TRANSLATIONAL SHELLS WITH A PURE-MOMENT STRESSED-STRAINED STATE

T. M. Martynenko

UDC 539.3:531.2.001:621.81

Consideration is given to the problem on determination of the shape of a shell with its median translational surface in which a prescribed force field (external load) generates a pure-moment stressed-strained state.

Thin-walled, three-dimensional structural elements of small thickness (shells) are widely used in engineering and construction, since, in using them, one can solve problems on reduction of the specific quantity of metal and material of the main product with preservation of a high strength, reliability, and longevity. Therefore, improvement of their analysis for strength, vibrations, and stability represents a topical problem of the modern mechanics of a deformed rigid body. The arising partial problems can arbitrarily be subdivided into primal problems and inverse problems. The first problems may include those of calculation of the stressed-strained state (SSS) of shells of prescribed geometric shape and external load, whereas the second class of problems may include determination of the geometric shape from prescribed SSS criteria and the law of external loading or straining. The condition of zero-moment (membrane) straining of the shell or the condition of its pure-moment straining by a prescribed load may act as the latter. An example of a zero-moment SSS is $\chi_1 = \chi_2 = \chi_{12} = 0$ (absence of bending moments and torques) and that of a pure-moment SSS is $\varepsilon_1 = \varepsilon_2 = \gamma_{12} = 0$ (absence of membrane forces in the shell). The focus of the present work is the second of the inverse problems enumerated above; it is solved within the framework of the Kirchhoff–Love theory of flat shells. Selection of this theory as the basic one is attributed to the simplicity of the basic hypotheses and equations, which reduce determination of the SSS of a shell to finding the strained state of its median surface (i.e., the surface dividing the thickness h in two).

We consider flat shells of constant thickness, whose median surfaces are described by equations of the form [1]

$$z = \varphi(\alpha) + \psi(\beta) , \quad \alpha \in [0, a] , \quad \beta \in [0, b] . \tag{1}$$

We assume for $\varphi(\alpha)$ and $\psi(\beta)$ that

$$A \approx 1 , \quad B \approx 1 \tag{2}$$

or

$$|\varphi'(\alpha)| << 1 , \quad |\psi'(\beta)| << 1 .$$

Then we have [1]

$$\frac{1}{R_1} \approx -\varphi''(\alpha) , \quad \frac{1}{R_2} \approx -\psi''(\beta) , \quad \frac{1}{R_{12}} = 0 . \tag{3}$$

Command-Engineering Institute of the Ministry of Emergencies of the Republic of Belarus, Minsk, Belarus.
Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 78, No. 3, pp. 145–149, May–June, 2005. Original article submitted February 16, 2004; revision submitted March 18, 2004.

It is necessary to determine the functions $\varphi(\alpha)$ and $\psi(\beta)$ from the conditions that, in the shell, a prescribed external load generates only moment stresses, i.e.,

$$T_1 = T_2 = S = 0\,, \quad T_{12} = \frac{H}{R_2}\,, \quad T_{21} = \frac{H}{R_1}\,; \tag{4}$$

$$M_1 = D\,(\chi_1 + \mu\chi_2)\,, \quad M_2 = D\,(\chi_2 + \mu\chi_1)\,, \quad H = D\,(1-\mu)\,\chi_{12}\,, \quad D = \frac{Eh^3}{12\,(1-\mu^2)}\,. \tag{5}$$

The resolving system of equations of the problem in question consists of the equilibrium equations, the equations of consistency of strains, and Hooke's law, which, by virtue of (2) and (3), take the form [1]

$$\frac{\partial}{\partial\beta}\left(\frac{H}{R_1}\right) + \frac{Q_1}{R_1} + q_1 = 0\,, \quad \frac{\partial}{\partial\alpha}\left(\frac{H}{R_2}\right) + \frac{Q_2}{R_2} + q_2 = 0\,, \quad \frac{\partial Q_1}{\partial\alpha} + \frac{\partial Q_2}{\partial\beta} + q_3 = 0\,; \tag{6}$$

$$\begin{gathered}\frac{\partial M_1}{\partial\alpha} + \frac{\partial H}{\partial\beta} - Q_1 = 0\,, \quad \frac{\partial M_2}{\partial\beta} + \frac{\partial H}{\partial\alpha} - Q_2 = 0\,,\\ \frac{\partial\chi_1}{\partial\beta} - \frac{\partial\chi_{12}}{\partial\alpha} = 0\,, \quad \frac{\partial\chi_2}{\partial\alpha} - \frac{\partial\chi_{12}}{\partial\beta} = 0\,, \quad \frac{\chi_1}{R_2} + \frac{\chi_2}{R_1} = 0\,;\end{gathered} \tag{7}$$

$$\chi_1 = \frac{12}{Eh^3}\,(M_1 - \mu M_2)\,, \quad \chi_2 = \frac{12}{Eh^3}\,(M_2 - \mu M_1)\,, \quad \chi_{12} = \frac{12\,(1+\mu)}{Eh^3}\,H\,. \tag{8}$$

Eliminating Q_1 and Q_2 from (6), we rewrite the equilibrium equations:

$$\begin{gathered}\frac{1}{R_1}\left(\frac{\partial M_1}{\partial\alpha} + 2\,\frac{\partial H}{\partial\beta}\right) + q_1 = 0\,, \quad \frac{1}{R_2}\left(\frac{\partial M_2}{\partial\beta} + 2\,\frac{\partial H}{\partial\alpha}\right) + q_2 = 0\,,\\ \frac{\partial}{\partial\alpha}\left(\frac{\partial M_1}{\partial\alpha} + \frac{\partial H}{\partial\beta}\right) + \frac{\partial}{\partial\beta}\left(\frac{\partial M_2}{\partial\beta} + \frac{\partial H}{\partial\alpha}\right) + q_3 = 0\,.\end{gathered} \tag{9}$$

From (9) we have

$$2\,\frac{\partial^2 H}{\partial\alpha\partial\beta} = q_3 - \frac{\partial}{\partial\alpha}\,(R_1 q_1) - \frac{\partial}{\partial\beta}\,(R_2 q_2)\,, \tag{10}$$

whence we have

$$2H\,(\alpha, \beta) = f\,(\alpha) + g\,(\beta) + \int_0^\alpha\!\int_0^\beta \left(q_3 - \frac{\partial}{\partial\alpha}\,(R_1 q_1) - \frac{\partial}{\partial\beta}\,(R_2 q_2)\right) d\alpha d\beta\,,$$

where $f(\alpha) = 2H(\alpha, 0) - g(0)$ and $g(\beta) = 2H(0, \beta) - f(0)$. Therefore, we have

$$f\,(\alpha) + g\,(\beta) = 2H\,(\alpha, 0) + 2H\,(0, \beta) - f\,(0) - g\,(0) = 2H\,(\alpha, 0) + 2H\,(0, \beta) - 2H\,(0, 0)\,.$$

Then we obtain

$$2H(\alpha, \beta) = 2H(\alpha, 0) + 2H(0, \beta) - 2H(0, 0) + \int_0^\alpha \int_0^\beta \left(q_3 - \frac{\partial}{\partial \alpha}(R_1 q_1) - \frac{\partial}{\partial \beta}(R_2 q_2) \right) d\alpha d\beta \,. \tag{11}$$

Here $H(\alpha, 0)$, $H(0, \beta)$, and $H(0, 0)$ are determined from the boundary conditions. Therefore, in what follows, $H(\alpha, \beta)$ will be considered to be the known function of $\alpha \in [0; a]$ and $\beta \in [0; b]$.

To find unknown M_1 and M_2 we use the first two of the equilibrium equations (9) and the equations of consistency of strains, which will be rewritten as

$$\frac{\partial M_1}{\partial \alpha} = -R_1 q_1 - 2\frac{\partial H}{\partial \beta}, \quad \frac{\partial M_2}{\partial \beta} = -R_2 q_2 - 2\frac{\partial H}{\partial \alpha},$$

$$\frac{\partial M_1}{\partial \beta} = (1+\mu)\frac{\partial H}{\partial \alpha} + \mu \frac{\partial M_2}{\partial \beta}, \quad \frac{\partial M_2}{\partial \alpha} = (1+\mu)\frac{\partial H}{\partial \beta} + \mu \frac{\partial M_1}{\partial \alpha}\,. \tag{12}$$

We can transform system (12) as

$$\frac{\partial M_1}{\partial \alpha} = -R_1 q_1 - 2\frac{\partial H}{\partial \beta}, \quad \frac{\partial M_2}{\partial \beta} = -R_2 q_2 - 2\frac{\partial H}{\partial \alpha},$$

$$\frac{\partial M_1}{\partial \beta} = (1-\mu)\frac{\partial H}{\partial \alpha} - \mu R_2 q_2, \quad \frac{\partial M_2}{\partial \alpha} = (1-\mu)\frac{\partial H}{\partial \beta} - \mu R_1 q_1\,. \tag{13}$$

We assume that we have the following equalities [2]:

$$\begin{gathered} -\frac{\partial}{\partial \beta}\left(R_1 q_1 + 2\frac{\partial H}{\partial \beta}\right) = \frac{\partial}{\partial \alpha}\left((1-\mu)\frac{\partial H}{\partial \alpha} - \mu R_2 q_2\right), \\ -\frac{\partial}{\partial \alpha}\left(R_2 q_2 + 2\frac{\partial H}{\partial \alpha}\right) = \frac{\partial}{\partial \beta}\left((1-\mu)\frac{\partial H}{\partial \beta} - \mu R_1 q_1\right). \end{gathered} \tag{14}$$

Then we obtain

$$\begin{gathered} M_1 = \int_{M_0 M} \left(-\left(R_1 q_1 + 2\frac{\partial H}{\partial \beta}\right)\right) d\alpha + \left((1-\mu)\frac{\partial H}{\partial \alpha} - \mu R_2 q_2\right) d\beta\,, \\ M_2 = \int_{M_0 M} \left((1-\mu)\frac{\partial H}{\partial \beta} - \mu R_1 q_1\right) d\alpha - \left(R_2 q_2 + 2\frac{\partial H}{\partial \alpha}\right) d\beta\,. \end{gathered} \tag{15}$$

The curvilinear integrals appearing in (15) are independent of the method of integration, since the integrands in them are represented by the total differentials when (14) holds.

The integro-differential equation sought is obtained using the third equation of system (7); we pretransform this equation, using (8):

$$\psi''(\beta)(M_1 - \mu M_2) + \varphi''(\alpha)(M_2 - \mu M_1) = 0\,. \tag{16}$$

Substituting (15) into (16), we finally obtain

$$\psi''(\beta)\left(\int_{M_0 M} \left(-\left(R_1 q_1 + 2\frac{\partial H}{\partial \beta}\right)\right) d\alpha + \left((1-\mu)\frac{\partial H}{\partial \alpha} - \mu R_2 q_2\right) d\beta - \right.$$

$$-\mu\int\limits_{M_0M}\left((1-\mu)\frac{\partial H}{\partial\beta}-\mu R_1q_1\right)d\alpha-\left(R_2q_2+2\frac{\partial H}{\partial\alpha}\right)d\beta\Bigg)+$$

$$+\varphi''(\alpha)\Bigg(\int\limits_{M_0M}\left((1-\mu)\frac{\partial H}{\partial\beta}-\mu R_1q_1\right)d\alpha-\left(R_2q_2+2\frac{\partial H}{\partial\alpha}\right)d\beta-$$

$$-\mu\int\limits_{M_0M}\left(-\left(R_1q_1+2\frac{\partial H}{\partial\beta}\right)\right)d\alpha+\left((1-\mu)\frac{\partial H}{\partial\alpha}-\mu R_2q_2\right)d\beta\Bigg)=0\,. \qquad (17)$$

The function $H = H(\alpha, \beta)$ involved in (17) is determined by formula (11). Equalities (14) should be considered as the conditions of correct solvability of Eq. (17) and hence the entire problem formulated.

We consider an example. Let

$$q_1=q_2=0\,,\quad q_3=\text{const}\,;\quad f(\alpha)=2H(\alpha,0)\,,\quad g(\beta)=2H(0,\beta)\,. \qquad (18)$$

In this case we have

$$2H(\alpha,\beta)=f(\alpha)+g(\beta)+q_3\alpha\beta\,,\qquad 2\frac{\partial H}{\partial\alpha}=f'(\alpha)+q_3\beta\,,\quad 2\frac{\partial H}{\partial\beta}=g'(\beta)+q_3\alpha\,,$$

$$2\frac{\partial^2H}{\partial\alpha^2}=f''(\alpha)\,,\quad 2\frac{\partial^2H}{\partial\beta^2}=g''(\beta)\,,\qquad \frac{1}{R_1}\approx-\varphi''(\alpha)\,,\quad \frac{1}{R_2}\approx-\psi''(\beta)\,. \qquad (19)$$

From (14) we have

$$-2\frac{\partial^2H}{\partial\beta^2}=(1-\mu)\frac{\partial^2H}{\partial\alpha^2}\,,\quad -2\frac{\partial^2H}{\partial\alpha^2}=(1-\mu)\frac{\partial^2H}{\partial\beta^2}\,. \qquad (20)$$

Into (20) we substitute

$$-2f''(\alpha)=(1-\mu)\,g''(\beta)\,,\quad -2g''(\beta)=(1-\mu)\,f''(\alpha)$$

or

$$2f''(\alpha)+(1-\mu)\,g''(\beta)=0\,,\quad 2g''(\beta)+(1-\mu)\,f''(\alpha)=0\,. \qquad (21)$$

Expression (21) is a homogeneous system of algebraic equations for $f''(\alpha)$ and $g''(\beta)$. Its determinant is equal to

$$\Delta=\begin{vmatrix}2 & 1-\mu\\ 1-\mu & 2\end{vmatrix}=4-(1-\mu)^2=(1+\mu)(3-\mu)\neq 0\,,\ \text{ since }\ \mu\in\left(0;\frac{1}{2}\right).$$

Therefore, we have

$$f''(\alpha)=0\ \text{ and }\ g''(\beta)=0\,,$$

consequently,

$$f'(\alpha)=c_1\,,\quad f(\alpha)=c_1\alpha+c_2\,,\quad g'(\beta)=c_3\,,\quad g(\beta)=c_3\beta+c_4\,. \qquad (22)$$

Whence, by virtue of (18), (21), and (17), we obtain

$$2H(\alpha, \beta) = q_3\alpha\beta + c_1\alpha + c_3\beta + c\,. \tag{23}$$

Expressions (15), (18), and (22) yield

$$M_1 = \int_{M_0M}\left(-2\,\frac{\partial H}{\partial \beta}\,d\alpha + (1-\mu)\,\frac{\partial H}{\partial \alpha}\,d\beta\right) = \int_{M_0M} -(q_3\alpha + c_3)\,d\alpha + \left(\frac{1-\mu}{2}\right)(q_3\beta + c_1)\,d\beta =$$

$$= \int_0^{\alpha} -(q_3\alpha + c_3)\,d\alpha + \int_0^{\beta}\left(\frac{1-\mu}{2}\right)(q_3\beta + c_1)\,d\beta = \frac{-2\alpha^2 + (1-\mu)\,\beta^2}{4}\,q_3 - c_3\alpha + \frac{1-\mu}{2}\,c_1\beta\,, \tag{24}$$

$$M_2 = \int_{M_0M}\left((1-\mu)\,\frac{\partial H}{\partial \beta}\,d\alpha - 2\,\frac{\partial H}{\partial \alpha}\,d\beta\right) = \int_{M_0M}\left(\frac{1-\mu}{2}\right)(q_3\alpha + c_3)\,d\alpha - (q_3\beta + c_1)\,d\beta =$$

$$= \int_0^{\alpha}\left(\frac{1-\mu}{2}\right)(q_3\alpha + c_3)\,d\alpha + \int_0^{\beta} -(q_3\beta + c_1)\,d\beta = \frac{-2\beta^2 + (1-\mu)\,\alpha^2}{4}\,q_3 - c_1\beta + \left(\frac{1-\mu}{2}\right)c_3\alpha\,. \tag{25}$$

Substituting (24) and (25) into (16), we obtain

$$\psi''(\beta)\left(\frac{-2\alpha^2 + (1-\mu)\,\beta^2}{4}\,q_3 - c_3\alpha + \frac{1-\mu}{2}\,c_1\beta - \mu\left(\frac{-2\beta^2 + (1-\mu)\,\alpha^2}{4}\,q_3 - c_1\beta + \frac{1-\mu}{2}\,c_3\alpha\right)\right) +$$

$$+ \varphi''(\alpha)\left(\frac{-2\beta^2 + (1-\mu)\,\alpha^2}{4}\,q_3 - c_1\beta + \frac{1-\mu}{2}\,c_3\alpha - \mu\left(\frac{-2\alpha^2 + (1-\mu)\,\beta^2}{4}\,q_3 - c_3\alpha + \frac{1-\mu}{2}\,c_1\beta\right)\right) = 0\,. \tag{26}$$

To determine the sought surface (1) we first set $\alpha = 0$ and $\beta \neq 0$ in (26) and then $\alpha \neq 0$ and $\beta = 0$. Then we obtain

$$\psi''(\beta)\left(\frac{1-\mu}{4}\,\beta^2 q_3 + \frac{1-\mu}{2}\,c_1\beta + \mu\left(\frac{1}{2}\,\beta^2 q_3 + c_1\beta\right)\right) -$$

$$-\,\varphi''(0)\left(\frac{1}{2}\,\beta^2 q_3 + c_1\beta + \mu\left(\frac{1-\mu}{4}\,\beta^2 q_3 + \frac{1-\mu}{2}\,c_1\beta\right)\right) = 0\,,$$

$$\varphi''(\alpha)\left(\frac{1-\mu}{4}\,\alpha^2 q_3 + \frac{1-\mu}{2}\,c_3\alpha + \mu\left(\frac{1}{2}\,\alpha^2 q_3 + c_3\alpha\right)\right) -$$

$$-\,\psi''(0)\left(\frac{1}{2}\,\alpha^2 q_3 + c_3\alpha + \mu\left(\frac{1-\mu}{4}\,\alpha^2 q_3 + \frac{1-\mu}{2}\,c_3\alpha\right)\right) = 0\,, \tag{27}$$

whence we have

$$\psi''(\beta)\left(\frac{1+\mu}{4}\,\beta^2 q_3 + \frac{1+\mu}{2}\,c_1\beta\right) = \varphi''(0)\left(\frac{2+\mu\,(1-\mu)}{4}\,\beta^2 q_3 + \frac{2+\mu\,(1-\mu)}{2}\,c_1\beta\right) \equiv$$

$$\equiv \varphi''(0)\left(-\,\frac{(\mu+1)\,(\mu-2)}{4}\,(\beta^2 q_3 + 2c_1\beta)\right),$$

$$\varphi''(\beta)\left(\frac{1+\mu}{4}\,\alpha^2 q_3 + \frac{1+\mu}{2}\,c_3\alpha\right) = \psi''(0)\left(\frac{2+\mu\,(1-\mu)}{4}\,\alpha^2 q_3 + \frac{2+\mu\,(1-\mu)}{2}\,c_3\alpha\right) \equiv$$

$$\equiv \psi''(0)\left(-\frac{(\mu+1)(\mu-2)}{4}(\alpha^2 q_3 + 2c_3\alpha)\right)$$

or

$$\psi''(\beta) = (2-\mu)\,\varphi''(0)\,, \quad \varphi''(\alpha) = (2-\mu)\,\psi''(0)\,. \tag{28}$$

Consequently,

$$\psi(\beta) = \frac{(2-\mu)}{2}\beta^2\varphi''(0) + A_1\beta + A_2\,, \quad \varphi(\alpha) = \frac{2-\mu}{2}\alpha^2\psi''(0) + B_1\alpha + B_2\,, \tag{29}$$

whence we find $\psi(0)$ and $\varphi(0)$:

$$\psi(0) = A_2\,, \quad \varphi(0) = B_2\,; \quad \psi'(0) = A_1\,, \quad \varphi'(0) = B_1\,.$$

Therefore, we obtain

$$\psi(\beta) = \left(\frac{(2-\mu)}{2}\beta^2 + 1\right)A_2 + A_1\beta\,, \quad \varphi(\alpha) = \left(\frac{2-\mu}{2}\alpha^2 + 1\right)B_2 + B_1\alpha\,, \tag{30}$$

where A_i and B_i (i = 1 and 2) are the integration constants. The median surface sought is described by the equation

$$z = \left(\frac{(2-\mu)}{2}\beta^2 + 1\right)A_2 + A_1\beta + \left(\frac{2-\mu}{2}\alpha^2 + 1\right)B_2 + B_1\alpha\,. \tag{31}$$

In deriving formula (31), it has been assumed that

$$\frac{1+\mu}{4}(\alpha^2 q_3 + 2c_3\alpha) \neq 0\,, \quad \frac{1+\mu}{4}(\beta^2 q_3 + 2c_1\beta) \neq 0\,,$$

$$\text{ò. å.} \quad \alpha \neq -\frac{2c_3}{q_3}\,, \quad \beta \neq 0\,; \quad \alpha \neq 0\,, \quad \beta \neq -\frac{2c_1}{q_1}\,. \tag{32}$$

NOTATION

A and *B*, coefficients of the first quadratic form; *E*, Young modulus; *h*, shell thickness; $1/R_1$, $1/R_2$, and $1/R_{12}$, curvatures and torsion of the median surface; $q_i(\alpha, \beta)$, components of the external surface load, $i = \overline{1, 3}$; μ, Poisson coefficient.

REFERENCES

1. V. L. Biderman, *Mechanics of Thin-Walled Structures* [in Russian], Mashinostroenie, Moscow (1977).
2. A. L. Gol'denveizer, *The Theory of Elastic Thin Shells* [in Russian], GITTL, Moscow (1953).